Consistent Discretization of Finite-time Stable Homogeneous Systems

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***Abstract*— An algorithm of implicit discretization for generalized homogeneous system having discontinuity only at the origin is developed. It is based on transformation of the original system to an equivalent standard homogeneous system which admits implicit discretization preserving finite-time convergence property. The scheme is demonstrated for a version of the so-called ”quasi-continuous” high-order sliding mode algorithm.**

I. INTRODUCTION

Symmetry is one of well-known properties of physical systems. A form of symmetry studied in systems and control theory is homogeneity [1], [2], [3]. The standard homogeneity introduced by L. Euler in 17th century is the symmetry of a mathematical object f (e.g. function, vector field, operator, etc) with respect to the uniform (or standard) *dilation* of the argument $x \to \lambda x$, namely, $f(\lambda x) = \lambda f(x), \lambda > 0$. Types of homogeneity are identified by the corresponding dilations. In [4], [5], [6] the uniform dilation is utilized but the papers [1], [7], [8] deal with the so-called weighted dilation. Nonlinear homogeneous differential equations/inclusions form an important class of control systems [9], [10], [11], [12]. They appear as local approximations [2], [12] or set-valued extensions [7], [8] of nonlinear systems and include models of process control [13], nonholonomic systems [14], mechanical models with frictions [7], etc.

The generalized homogeneity (to be considered in this paper) was introduced originally in [15] for infinite dimensional models such as partial differential equations (PDEs). It considers a strongly continuous group of linear bounded operators generated by a possibly unbounded linear operator as a *dilation* in a Banach space. A lot of well-known PDEs are homogeneous in generalized sense, e.g. heat, wave, Navier-Stocks, Saint-Venant, Korteweg-de Vries, fast diffusion equations. This paper deals with the finite-dimensional models of generalized homogeneous systems represented by ordinary differential equations (ODEs). In [16] it has been proved that any generalized homogeneous system is topologically equivalent (homeomorphic on $\mathbb{R}^n$ and diffeomorphic on $\mathbb{R}^n\backslash\{\mathbf{0}\}$) to standard homogeneous one, but any asymptotically stable homogeneous system is topologically equivalent to a quadratically stable system. These two facts are essentially used in the present paper.

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Discretization issues are very important for digital implementation of estimation and control algorithms. The sliding mode algorithms are known to be difficult in practical realization due to discontinuous (set-valued) nature, which may invoke chattering caused by discretization. Finite-time stability is a desirable property of sliding mode algorithms [17], [7], [8]. Usually [18], [19] discretization destroys finite-time convergence property. However, for 1-order and 2-order (”twisting”) algorithms implicit discretization approach allows finite-time stability to be preserved for the discretized system [20], [21]. In this paper such an implicit scheme is developed for homogeneous finite-time stable systems.

A. Motivating Example

Let us consider two scalar non-linear control systems:

$$\begin{array}{lcl} \dot{x} = u(x) & \overset{y=\sqrt{|x|}\mathrm{sgn}(x)}{\Leftrightarrow} & \dot{y} = \tilde{u}(y), \\ u(x) = -2\sqrt{|x|}\mathrm{sgn}(x), & & \tilde{u}(y) \in -\mathrm{sgn}(y), \end{array}$$

where

$$\mathrm{sgn}(\rho) = \begin{cases} 1 & \text{if} \quad \rho > 0, \\ [-1, 1] & \text{if} \quad \rho = 0, \\ -1 & \text{if} \quad \rho < 0. \end{cases}$$

Both systems are finite-time stable, i.e. the state of each system vanishes in a finite time. These two systems are topologically equivalent (homeomorphic on $\mathbb{R}$ and diffeomorphic on $\mathbb{R}\backslash\{\mathbf{0}\}$). More precisely, if $x(\cdot, x_0)$ is the solution of the first system with $x(0) = x_0 \in \mathbb{R}$ then $y(\cdot, y_0) = \sqrt{|x(\cdot, x_0)|}\mathrm{sgn}(x(\cdot, x_0))$ is the solution of the second system with $y(0) = y_0 = \sqrt{|x_0|}\mathrm{sgn}(x_0)$, and vise a versa.

Both control systems admit an implicit discretization (sampled-time realization) of the control:

$$\begin{array}{lcl} x_{n+1} = x_n + h u_{n+1} & \not\Leftrightarrow & y_{n+1} = y_n + h\tilde{u}_{n+1} \\ u_{n+1} = -2\sqrt{|x_{n+1}|}\mathrm{sgn}(x_{n+1}) & & \tilde{u}_{n+1} \in \mathrm{sgn}(y_{n+1}) \end{array}$$

where h is the sampling period, $n = 0, 1, 2...$ and $x_{n+1} = x((n+1)h, x_0)$ (resp. $y_{n+1} = y((n+1)h, x_0)$) provided that $u(t) = u_{n+1}$ (resp. $\tilde{u}(t) = \tilde{u}_{n+1}$) for $t \in [nh, (n+1)h]$.

The topological equivalence between the two systems is broken after discretization. Indeed, for the first system one has

$$\begin{array}{ll} x_{n+1} = \left(\sqrt{h^2 + |x_n|} - h\right)^2 \mathrm{sgn}(x_0) \neq 0, & \forall n \geq 0 \\ u_{n+1} = 2(h - \sqrt{h^2 + |x_n|})\mathrm{sgn}(x_0), & \end{array}$$

provided that $x_0 \neq 0$, but the implicit scheme for the second equation gives (see, [21] for the details)

$$y_{n+1} = \begin{cases} y_n - h\,\mathrm{sgn}(y_n) & \text{if} \quad |y_n| > h, \\ 0 & \text{if} \quad |y_n| \leq h. \end{cases}$$

$$u_{n+1} = \begin{cases} -\mathrm{sgn}(y_n) & \text{if} \quad |y_n| > h, \\ -\frac{y_n}{h} & \text{if} \quad |y_n| \leq h. \end{cases}$$

In other words, the implicit discretization of the first equation is just asymptotically stable, but the implicit discretization of the second equation remains finite-time stable, i.e. there exists n^* dependent of y_0 and h such that $y_n = 0$ for all $n \geq n^*$.

This example motivates us for the conjecture that even continuous finite-time stable system may have *a consistent discretization* that preserves the finite-time stability property. A design of the corresponding discretization scheme is expected to be based on transformation of the original system to an equivalent one, which *implicit discretization* is finite-time stable. In this paper the design procedure is given for a class of generalized homogeneous systems, which includes, as particular cases, some "quasi-continuous" high-order sliding mode algorithms [8], [22].

B. Notation

$\mathbb{R}$ is the field of real numbers; $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; $\|\cdot\|$ denotes a norm in $\mathbb{R}^n$ and

$$\|A\|_{\mathbb{A}} = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \lfloor A \rfloor_{\mathbb{A}} = \inf_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|} \text{ if } A\in\mathbb{R}^{n\times n};$$

$C^n(X,Y)$ is the set of continuously differentiable (at least up to the order n) maps $X \to Y$, where X, Y are open subsets of finite dimensional spaces; $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$; $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix; $\mathbf{0}$ denotes zero element, e.g. $\mathbf{0} \in \mathbb{R}^n$ is the zero vector but $\mathbf{0} \in \mathbb{R}^{n\times n}$ is the zero matrix; $\mathrm{diag}\{\lambda_1,..,\lambda_n\}$ - diagonal matrix with elements λ_i; the order relation $P \succ 0$ means positive definiteness of the symmetric matrix $P \in \mathbb{R}^{n\times n}$; $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote maximal and minimal eigenvalues of the symmetric matrix $P \in \mathbb{R}^{n\times n}$; $\Re(\lambda)$ denotes the real part of the complex number λ; the notation $P^{\frac{1}{2}}$ means that $P^{\frac{1}{2}} = M$ is such that $P = M^2$; a function $c : [0, +\infty) \to [0, +\infty)$ belongs to the class $\mathcal{K}$ if it is continuous, monotone increasing and $c(0) = 0$.

II. Problem Statement

Let us consider the non-linear Cauchy problem

$$\dot{x}(t) = f(x(t)), \quad t > 0, \quad x(0) = x_0 \neq \mathbf{0}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n\backslash\{\mathbf{0}\}$, i.e. ***the only possible discontinuity point of f is the origin***. The system (1) is assumed to be forward complete for solutions understood in the sense of Filippov [23]: *an absolutely continuous function $\phi(\cdot, x_0)$: $[0, +\infty) \to \mathbb{R}^n$ is a solution to* (1) *if $\phi(0, x_0) = x_0$ and for almost all $t > 0$ it satisfies the differential inclusion*

$$\dot{x}(t) \in F(x(t)) = \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} cof(x + \varepsilon B\backslash\{N\}), \tag{2}$$

where B denotes the unit ball in $\mathbb{R}^n$ and $\mu(N)$ means that the Lebesgue measure of the set $N \subset \mathbb{R}^n$ is zero. Obviously, in the general case, the Filippov solution to (1) is not unique.

In our case, $F(x) = \{f(x)\}$ is a singleton for $x \in \mathbb{R}^n\backslash\{\mathbf{0}\}$, but $F(\mathbf{0})$ is set-valued if f is discontinuous at $\mathbf{0}$.

Assumption 1: The origin of the system (1) *is globally uniformly finite-time stable.*

The latter means that *the origin is Lyapunov stable and there exists a locally bounded function $T : \mathbb{R}^n \to [0, +\infty)$ such that any solution $\phi(\cdot, x_0)$ to* (1) *satisfies $\phi(t, x_0) = 0$ for $t \geq T(x_0)$.*

*Definition 1: A set-valued map $Q : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be a **consistent** discrete-time approximation of the globally uniformly finite-time stable system* (1) *if*

- ***Existence property**: for any $\tilde{x} \in \mathbb{R}^n$ and any $h > 0$ there exists $\tilde{x}_h \in \mathbb{R}^n$:*

$$\mathbf{0} \in Q(h, \tilde{x}, \tilde{x}_h); \tag{3}$$

  and $\tilde{x}_h = \mathbf{0}$ is the unique solution to $\mathbf{0} \in Q(h, \mathbf{0}, \tilde{x}_h)$;
- ***Finite-time convergence property:** for any fixed $h > 0$ each sequence*

$$\{x_i\}_{i=0}^{+\infty}, \quad x_0 \neq \mathbf{0} \tag{4}$$

  generated by

$$\mathbf{0} \in Q(h, x_i, x_{i+1}), \quad i = 0, 1, 2, ... \tag{5}$$

  converges to zero in a finite number of steps, i.e. there exists $i^* > 0$ such that

$$x_j = \mathbf{0} \qquad \textit{for} \quad j \geq i^*$$

  and $x_{i^*-1} \neq \mathbf{0}$;
- ***Approximation property**: for any $\varepsilon > 0$ there exist $\gamma, \omega \in \mathcal{K}$ such that for any $x_0 \in \mathbb{R}^n : \|x_0\| > \varepsilon$ and for any $h > 0$ the sequence* (4) *generated by* (5) *satisfies*

$$\|\phi(h, x_{i-1}) - x_i\| \leq h(1+\gamma(\|x_0\|))\, \omega(h), \quad i < i^* \tag{6}$$

  where $\phi(\cdot, x_{i-1})$ is a solution (1) *with the intial condition $x(0) = x_{i-1}$.*

Since the inequality (6) describes local (one-step) approximation error then the approximation error on the interval $[0, T(x_0)]$ is $O((1 + \gamma(\|x_0\|))\, \omega(h))$. Obviously, this error tends to zero as $h \to 0$.

The aim of the paper is to design a consistent (in the sense of Definition 1) implicit discretization scheme for the finite-time stable system (1) under the assumption that the vector field f is homogeneous in a generalized sense [16]. Most of high-order sliding mode algorithms are homogeneous [8] or locally homogeneous.

III. Preliminaries

A. Generalized Homogeneity

The generalized homogeneity [15] deals with the group of linear transformations (*linear dilations*).

*Definition 2 ([15]): A map $\boldsymbol{d} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is called **dilation** in $\mathbb{R}^n$ if it satisfies*

- ***Group property:** $\boldsymbol{d}(0) = I_n$ and $\boldsymbol{d}(t+s) = \boldsymbol{d}(t)\boldsymbol{d}(s) = \boldsymbol{d}(s)\boldsymbol{d}(t)$, $t, s \in \mathbb{R}$;*
- ***Continuity property:** $\boldsymbol{d}$ is a continuous map, i.e.* $\forall t > 0, \forall \varepsilon > 0, \exists \delta > 0 \ : \ |s-t| < \delta \ \Rightarrow \ \|\boldsymbol{d}(s) - \boldsymbol{d}(t)\|_{\mathbb{A}} \leq \varepsilon$;
- ***Limit property:** $\lim\limits_{s\to-\infty} \|\boldsymbol{d}(s)x\| = 0$ and $\lim\limits_{s\to+\infty} \|\boldsymbol{d}(s)x\| = +\infty$ uniformly on the unit sphere*

$$S := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

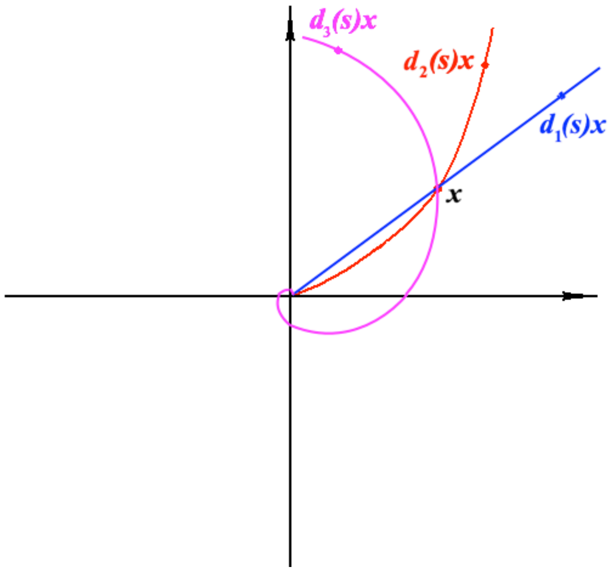

Fig. 1. Illustration of uniform $\mathbf{d}_1$, weighted $\mathbf{d}_2$ and generalized $\mathbf{d}_3$ dilations

The dilation $\mathbf{d}$ is a continuous group of invertible linear maps $\mathbf{d}(s) \in \mathbb{R}^{n\times n}$, $\mathbf{d}(-s) = [\mathbf{d}(s)]^{-1}$. The matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$

$$G_{\mathbf{d}} = \lim_{s\to 0} \tfrac{\mathbf{d}(s)-I}{s}$$

is known (see, e.g. [24, Ch. 1]) as the **generator** of the group $\mathbf{d}$. It satisfies the following properties

$$\tfrac{d\,\mathbf{d}(s)}{ds} = G_{\mathbf{d}}\mathbf{d}(s) \quad\text{and}\quad \mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \sum_{i=0}^{+\infty} \tfrac{s^i G_{\mathbf{d}}^i}{i!},\ s\in\mathbb{R}. \quad (7)$$

The latter implies $\mathbf{d}(s_1) - \mathbf{d}(s_2) = G_{\mathbf{d}} \int_{s_2}^{s_1} \mathbf{d}(s)ds$, $s_1, s_2 \in \mathbb{R}$. The most popular dilations in $\mathbb{R}^n$ are (see e.g. [8], [7])

- *uniform (or standard) dilation* (L. Euler 17th century) :
$$\mathbf{d}_1(s) = e^s I_n, \quad s\in\mathbb{R},$$
- *weighted dilation* (Zubov 1958, [1]):
$$\mathbf{d}_2(s) = \begin{pmatrix} e^{r_1 s} & 0 & \dots & 0 \\ 0 & e^{r_2 s} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{r_n s} \end{pmatrix},\ s\in\mathbb{R},\ r_i>0,\ i=1,...,n$$

They satisfy Definition 2 with $G_{\mathbf{d}_1} = I_n$ and $G_{\mathbf{d}_2} = \mathrm{diag}\{r_i\}$, respectively. Schematically the difference between uniform, weighted and generalized dilations is depicted at Figure 1.

Definition 3 ([15]): The dilation $\mathbf{d}$ *is said to be* ***strictly monotone*** *if* $\exists \beta > 0 : \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s}$ *as* $s<0$.

Obviously, monotonicity of dilation may depend on $\|\cdot\|$.

Theorem 1: [16] *Let* $\mathbf{d}$ *be a dilation in* $\mathbb{R}^n$ *then*

1) *all eigenvalues* λ_i *of the matrix* $G_{\mathbf{d}}$ *are placed in the right complex half-plane, i.e.* $\Re(\lambda_i) > 0, i = 1, 2, ..., n$;
2) *there exists a matrix* $P \in \mathbb{R}^{n\times n}$ *such that*
$$PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P = P^\top \succ 0; \quad (8)$$
3) *the dilation* $\mathbf{d}$ *is strictly monotone with respect to the weighted Euclidean norm* $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ *induced by the inner product* $\langle x, z\rangle = x^\top Pz$ *with* P *satisfying* (8)*:*
$$\begin{array}{l} e^{\alpha s} \leq \lfloor \mathbf{d}(s)\rfloor_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s} \ \ \text{if}\ \ s\leq 0, \\ e^{\beta s} \leq \lfloor \mathbf{d}(s)\rfloor_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\alpha s} \ \ \text{if}\ \ s\geq 0, \end{array} \quad (9)$$
where $\alpha = \frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right)$ *and* $\beta = \frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^\top P^{\frac{1}{2}}\right)$.

The latter theorem proves that any dilation $\mathbf{d}$ is strictly monotone if $\mathbb{R}^n$ is equipped with the norm $\|x\| = \sqrt{x^\top Px}$ provided that the matrix $P\succ 0$ satisfies (8).

Definition 4: A continuous function $p : \mathbb{R}^n \to \mathbb{R}_+$ *is said to be* $\mathbf{d}$*-homogeneous norm if* $p(x) \to 0$ *as* $x \to \mathbf{0}$ *and* $p(\mathbf{d}(s)x) = e^s p(x) > 0$ *for* $x \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ *and* $s \in \mathbb{R}$.

For monotone dilations the **canonical homogeneous norm** $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+$ is defined as

$$\|x\|_{\mathbf{d}} = e^{s_x} \quad \text{where}\ \ s_x \in \mathbb{R} \ \ : \ \ \|\mathbf{d}(-s_x)x\| = 1. \quad (10)$$

Proposition 1: [16] *If* $\mathbf{d}$ *is a strictly monotone dilation then*

- *the canonical homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *is continuous on* $\mathbb{R}^n$ *and Lipschitz continuous on* $\mathbb{R}^n\backslash\{\mathbf{0}\}$*;*
- *if the norm* $\|\cdot\|$ *is smooth outside the origin then the homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *is also smooth outside the origin,* $\frac{d\|\mathbf{d}(-s)x\|}{ds} < 0$ *if* $s\in\mathbb{R}$, $x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$ *and*

$$\frac{\partial\|x\|_{\mathbf{d}}}{\partial x} = \left.\frac{\|x\|_{\mathbf{d}} \left.\frac{\partial\|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x}}{\left.\frac{\partial\|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x} G_{\mathbf{d}}\mathbf{d}(-s)x}\right|_{s=\ln\|x\|_{\mathbf{d}}} \quad (11)$$

Below we use the notation $\|\cdot\|_{\mathbf{d}}$ only for the canonical homogeneous norm induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top Px}$ with a matrix $P\succ 0$ satisfying (8). The unit sphere S is defined using the same norm.

Vector fields, which are symmetric (homogeneous) in a certain sense with respect to dilation $\mathbf{d}$, have a lot of properties useful for control design and state estimation of both linear and nonlinear plants as well as for analysis of convergence rates [3], [25], [26], [11].

Definition 5: A vector field $f:\mathbb{R}^n\to\mathbb{R}^n$ *(resp. a function* $h:\mathbb{R}^n\to\mathbb{R}$*) is said to be* $\mathbf{d}$*-homogeneous if there exists* $\nu \in \mathbb{R}$

$$f(\mathbf{d}(s)x) = e^{\nu s}\mathbf{d}(s)f(x),\ \forall x \in \mathbb{R}^n\backslash\{\mathbf{0}\},\ \forall s \in \mathbb{R}. \quad (12)$$

$$\text{(resp. } h(\mathbf{d}(s)x) = e^{\nu s}h(x),\ \forall x\in\mathbb{R}^n\backslash\{\mathbf{0}\},\ \forall s\in\mathbb{R}.) \quad (13)$$

The number $\nu \in \mathbb{R}$ is called homogeneity degree.

Let $\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ (resp. $\mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$) be the set of vector fields $\mathbb{R}^n \to \mathbb{R}^n$ (resp. functions $\mathbb{R}^n \to \mathbb{R}$) satisfying the identity (12) (resp. (13)), which are **continuous on** $\mathbb{R}^n\backslash\{\mathbf{0}\}$. Let $\deg_{\mathbb{F}_{\mathbf{d}}}(f)$ (resp. $\deg_{\mathbb{H}_{\mathbf{d}}}(h)$) denote the homogeneity degree of $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ (resp. $h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$).

Proposition 2: If $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *is uniformly continuous*[1] *on the unit sphere* S *with the modulus of continuity* $\omega \in \mathcal{K}$*, then*

[1] A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be uniformly continuous on a set Ω if there exists $\omega \in \mathcal{K}$ such that $\|f(x_1) - f(x_2)\| \leq \omega(\|x_1 - x_2\|)$ for any $x_1, x_2 \in \Omega$, where the function ω is the so-called modulus of continuity.

it is uniformly continuous on any set

$$K(r_1,r_2)=\{x\in\mathbb{R}^n: r_1\le \|x\|_{\mathbf{d}}\le r_2\},\quad 0<r_1<1<r_2$$

with the modulus of continuity

$$\omega_{r_1,r_2}(\sigma)\le r_2^{\alpha+\nu}\omega\left(\tfrac{M}{r_1^{2\alpha-\beta}}\sigma\right)+M\max\left\{1,\left(\tfrac{r_1}{r_2}\right)^{\nu}\right\}\tfrac{r_2^{\alpha+\nu}}{r_1^{\alpha-\beta}}\sigma,$$

where $\nu=\deg_{\mathbb{F}_{\mathbf{d}}}(f)$*,* $M>0$ *is a constant number independent of* r_1 *and* r_2 *and* $0<\beta\le\alpha$ *are defined in Theorem 1.*

B. Quadratic Stability of Nonlinear Homogeneous Systems

Homogeneity may simplify the analysis of differential equations. The most important property of **d**-homogeneous systems is scalability of the solutions [1], [9], [10], [26].

Theorem 2 ([16]): If $\varphi_{\xi_0}:[0,T)\to\mathbb{R}^n$ *is a solution to*

$$\dot{\xi}=f(\xi),\quad f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n) \tag{14}$$

with the initial condition $\xi(0)=\xi_0\in\mathbb{R}^n$ *then* $\varphi_{\mathbf{d}(s)\xi_0}:[0,e^{-\nu s}T)\to\mathbb{R}^n$ *defined as* $\varphi_{\mathbf{d}(s)\xi_0}(t):=\mathbf{d}(s)\varphi_{\xi_0}(te^{\nu s})$ *with* $s\in\mathbb{R}$ *is a solution to* (14) *with the initial condition* $\xi(0)=\mathbf{d}(s)\xi_0$*, where* $\nu=\deg_{\mathbb{F}_{\mathbf{d}}}(f)$.

This theorem has a lot of corollaries. We use the next one.

Theorem 3 ([16]): The next five claims are equivalent

1) *The origin of the system* (14) *is asymptotically stable.*
2) *There exists Lyapunov function* $V\in\mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)\cap C^{\infty}(\mathbb{R}^n)$*;*
3) *The origin of the system*
$$\dot{z}=\|z\|^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f)}\left(\tfrac{(I_n-G_{\mathbf{d}})zz^{\top}P}{z^{\top}PG_{\mathbf{d}}z}+I_n\right)f\left(\tfrac{z}{\|z\|}\right) \tag{15}$$
is asymptotically stable.
4) *For any matrix* $P\in\mathbb{R}^{n\times n}$ *satisfying* (8) *there exists* $\Psi\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)\cap C^{\infty}(\mathbb{R}^n\backslash\{\mathbf{0}\})$, $\deg_{\mathbb{F}_{\mathbf{d}}}(\Psi)=0$ *such that* Ψ *is diffemorphism on* $\mathbb{R}^n\backslash\{\mathbf{0}\}$*, homeomorphism on* $\mathbb{R}^n$*,* $\Psi(\mathbf{0})=\mathbf{0}$ *and*
$$\tfrac{\partial\left(\Psi^{\top}(\xi)P\,\Psi(\xi)\right)}{\partial\xi}f(\xi)<0\quad if\quad \Psi^{\top}(\xi)P\Psi(\xi)=1. \tag{16}$$
Moreover, $\|\Psi\|_{\mathbf{d}}\in\mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)\cap C^{\infty}(\mathbb{R}^n\backslash\{\mathbf{0}\})$ *is Lyapunov function to the system* (14).
5) *For any matrix* $P\in\mathbb{R}^{n\times n}$ *satisfying* (8) *there exists a map* $\Xi\in C^{\infty}(\mathbb{R}^n\backslash\{\mathbf{0}\},\mathbb{R}^{n\times n})$ *such that*
$$\det(\Xi(z))\neq0,\quad \tfrac{\partial\Xi(z)}{\partial z_i}z=0,\quad \Xi(e^s z)=\Xi(z)$$
$$for\quad z=(z_1,...,z_n)^{\top}\in\mathbb{R}^n\backslash\{\mathbf{0}\},s\in\mathbb{R},i=1,...,n$$
and
$$z^{\top}\Xi^{\top}(z)P\Xi(z)\left(\tfrac{(I_n-G_{\mathbf{d}})zz^{\top}P}{z^{\top}PG_{\mathbf{d}}z}+I_n\right)f\left(\tfrac{z}{\|z\|}\right)<0. \tag{17}$$

This theorem proves two important facts:

- Any generalized homogeneous system (14) is homeomorphic on $\mathbb{R}^n$ and diffeomorphic on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ to a standard homogeneous one (15). The corresponding change of coordinates is given by
$$z=\|\xi\|_{\mathbf{d}}\mathbf{d}(-\ln\|\xi\|_{\mathbf{d}})\xi \tag{18}$$
while the inverse transformation is as follows
$$\xi=\mathbf{d}(\ln\|z\|)\frac{z}{\|z\|}$$
- Any asymptotically stable generalized homogeneous system is homeomorphic on $\mathbb{R}^n$ and diffeomorphic on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ to a **quadratically stable** one. Indeed, making the change of variables $z=\Psi(\xi)$ we derive
$$\dot{z}=f_0(z)=\left.\frac{\partial\Psi(\xi)}{\partial\xi}f(\xi)\right|_{\xi=\Psi^{-1}(z)},$$
but the criterion (16) implies that $z^{\top}P\dot{z}<0$ if $z^{\top}Pz=1$, so the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is the Lyapunov function to the latter system. Finally, the change of variable $x=\|z\|_{\mathbf{d}}\mathbf{d}(-\ln\|z\|_{\mathbf{d}})z$ gives $\|z\|_{\mathbf{d}}=\|x\|$, so the transformed system $\dot{x}=\tilde{f}(x)$ is quadratically stable with the Lyapunov function V defined as $V(x)=\|x\|^2=x^{\top}Px$, where $\tilde{f}(x)=\|x\|^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f)}\left(\frac{(I_n-G_{\mathbf{d}})xx^{\top}P}{x^{\top}PG_{\mathbf{d}}x}+I_n\right)f_0\left(\frac{x}{\|x\|}\right)$.

Lemma 1: If $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *is uniformly continuous on* S *then*

$$\tilde{f}(z)=\|z\|^{1+\deg_{\mathbb{F}_{\mathbf{d}}}(f)}\left(\tfrac{(I_n-G_{\mathbf{d}})z^{\top}zP}{z^{\top}PG_{\mathbf{d}}z}+I_n\right)f\left(\tfrac{z}{\sqrt{z^{\top}Pz}}\right)$$

is uniformly continuous S.

Recall [26], [7], [8], if the standard (or weighted) homogeneous system (14) is asymptotically stable and $\deg_{\mathbb{F}_{\mathbf{d}}}(f)<0$ then it is globally uniformly *finite-time stable*.

Remark 1: If $\mathbf{d}$ *is a dilation with the generator* $G_{\mathbf{d}}$ *then for any fixed* $\alpha>0$ *the group* $\mathbf{d}^{\alpha}$ *defined as* $\mathbf{d}^{\alpha}(s):=\mathbf{d}(\alpha s)$, $s\in\mathbb{R}^n$ *is the dilation with the generator* $G_{\mathbf{d}^{\alpha}}=\alpha G_{\mathbf{d}}$*. Moreover, if* $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *then* $f\in\mathbb{F}_{\mathbf{d}^{\alpha}}(\mathbb{R}^n)$ *and* $\deg_{\mathbb{F}_{\mathbf{d}^{\alpha}}}(f)=\alpha\deg_{\mathbb{F}_{\mathbf{d}}}(f)$*. In other words, if* $\deg_{\mathbb{F}_{\mathbf{d}}}(f)<0$ *then a new dilation* $\mathbf{d}^{\alpha}$ *can be selected such that* $\deg_{\mathbb{F}_{\mathbf{d}^{\alpha}}}(f)=-1$.

IV. Finite-time Stable Implicit Discretization

The main idea of the design of finite-time stable discretization for homogeneous system is to use the coordinate transformation (18). If $\deg_{\mathbb{F}}(f)=-1$ then the right hand side of the transformed system (15) will be globally bounded.

Theorem 4: Let $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ *be uniformly continuous on* S*,* $\deg_{\mathbb{F}_{\mathbf{d}}}(f)=-1$*,* $f(-x)=-f(x)$ *for* $x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$ *and*

$$\tilde{f}(z)=\left(\tfrac{(I-G_{\mathbf{d}})zz^{\top}P}{z^{\top}G_{\mathbf{d}}Pz}+I_n\right)f\left(\tfrac{z}{\sqrt{z^{\top}Pz}}\right),\quad z\in\mathbb{R}^n\backslash\{\mathbf{0}\},$$

where $G_{\mathbf{d}}$ *is the generator of the dilation* $\mathbf{d}$ *and the symmetric matrix* $P\in\mathbb{R}^{n\times n}$ *satisfies* (8)*. If the condition* (17) *holds with* Ξ *=const then the map* $Q:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ *defined as*

$$Q(h,\tilde{x},\tilde{x}_h)=\tilde{Q}(h,\Phi(\tilde{x}),\Phi(\tilde{x}_h)), \tag{19}$$

where $h>0,\tilde{x}\in\mathbb{R}^n,\tilde{x}_h\in\mathbb{R}^n$*,* $\Phi(\tilde{x})=\|\tilde{x}\|_{\mathbf{d}}\mathbf{d}(-\ln\|\tilde{x}\|_{\mathbf{d}})\tilde{x}$

$$\tilde{Q}(h,\tilde{y},\tilde{y}_h)=\tilde{y}_h-\tilde{y}-h\tilde{F}(\tilde{y}_h),\quad \tilde{y}\in\mathbb{R}^n,$$

$$\tilde{F}(\tilde{y}_h)=\bigcap_{\varepsilon>0}\bigcap_{\mu(N)=0}co\tilde{f}(\tilde{y}_h+\varepsilon B\backslash\{N\}),\quad \tilde{y}_h\in\mathbb{R}^n, \tag{20}$$

is the consistent discrete-time approximation of the finite-time stable system (1).

The latter theorem is based on the fact that the system $\dot{y}=\tilde{f}(y)$ admits a quadratic Lyapunov function. However, as it was shown in Theorem 3 any stable homogeneous system is equivalent to a quadratically stable system. Therefore, if f in Theorem 4 is replaced with

$$f^{new}(z) = \left.\frac{\partial\Psi(\xi)}{\partial\xi} f(\xi)\right|_{\xi=\Psi^{-1}(z)},$$

where Ψ is given in Theorem 3, then the condition $\Xi = const$ required for Theorem 4 is fulfilled for $\Xi = I_n$. Therefore, taking into account Remark 1 we conclude that

> *Any symmetric $f(-x) = -f(x), x \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ generalized homogeneous finite-time stable system admits a consistent implicit approximation.*

If $V \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n\backslash\{\mathbf{0}\})$ is a homogeneous Lyapunov function with $\deg_{\mathbb{H}}(V) = 1$ then the transformation Ψ mentioned above can be defined as [16]

$$\Psi(z) = \mathbf{d}\left(\ln \frac{V(x)}{\|x\|_{\mathbf{d}}}\right) x.$$

In other words, *if we know a homogeneous Lyapunov function for a finite-time stable homogeneous system then using Theorem 4 we can always design a consistent (in the sense of Definition 1) implicit discretization scheme.*

V. Example: Discrete-time Approximation of "Quasi-Continuous" Homogeneous Control System

Let us consider the system

$$\dot{x} = Ax + Bu(x), \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}, \quad (21)$$

where $x = (x_1, x_2, ..., x_n)^\top$,

$$u(x) = \sum_{j=1}^{n} k_j \frac{x_j}{N^{n-j+1}(x)}, \quad K = \begin{pmatrix} k_1 \\ k_2 \\ \dots \\ k_n \end{pmatrix} \in \mathbb{R}^n$$

and $N : \mathbb{R}^n \to \mathbb{R}_+$ is a $\mathbf{d}$-homogeneous norm and the dilation is defined as follows

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}}, \quad G_{\mathbf{d}} = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n-1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

Obviously, $u(\mathbf{d}(s)x) = u(x)$ and the closed-loop system is homogeneous of degree $\deg_{\mathbb{F}_{\mathbf{d}}} = -1$, i.e. u is a quasi-continuous sliding mode algorithm.

The equivalent system with $y = \|x\|_{\mathbf{d}}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$ has the form

$$\dot{y} = \tilde{f}(y) := \left(\frac{(I_n - G_{\mathbf{d}})yy^\top P}{y^\top PG_{\mathbf{d}}y} + I_n\right) g(y),$$

where

$$g(y) = \frac{1}{\|y\|}\begin{pmatrix} y_2 \\ \dots \\ y_{n-1} \\ \sum_{j=1}^{n} \frac{k_j y_j}{\tilde{N}^{n-j+1}(y)} \end{pmatrix}, \quad \tilde{N}(y) = N\left(\frac{y}{\|y\|}\right)$$

where $\|y\| = \sqrt{y^\top Py}$ with P satisfying (8). Note that if

$$N(x) = \sum_{j=1}^{n} q_i |x_j|^{\frac{1}{n-j+1}}, \quad q_j > 0,$$

then we derive a version of the "quasi-continuous" HOSM algorithm [22].

For simplicity we consider $N(x) = \|x\|_{\mathbf{d}}$, that was also studied in [27]. In this case we derive $\tilde{N}(y) = 1$.

Let the gain matrix K and a positive definite matrix P are selected as follows

$$(A+BK+G_{\mathbf{d}})^\top P + P(A+BK+G_{\mathbf{d}}) = 0, \quad PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0.$$

Such a selection is always possible (see, [28]).

In this case, we derive

$$\begin{aligned}\tilde{f}(y) &:= \left(\frac{(I_n - G_{\mathbf{d}})yy^\top P}{y^\top PG_{\mathbf{d}}y} + I_n\right)(A+BK)\frac{y}{\|y\|} = \\ &\frac{1}{\|y\|}\frac{(I_n - G_{\mathbf{d}})yy^\top P(A+BK)y}{y^\top PG_{\mathbf{d}}y} + (A+BK)\frac{y}{\|y\|} = \\ &\frac{1}{\|y\|}\frac{(I_n - G_{\mathbf{d}})y(-y^\top PG_{\mathbf{d}}y)}{y^\top PG_{\mathbf{d}}y} + (A+BK)\frac{y}{\|y\|} \\ &= (A + BK + G_{\mathbf{d}} - I_n)\frac{y}{\|y\|}.\end{aligned}$$

Therefore, the implicit discretization scheme has the very simple representation

$$y_i \in y_{i+1} + h\left(I_n - \tilde{A}\right)\tilde{F}(y_{i+1}), \quad h > 0, \ i = 0, 1, 2, ... \quad (22)$$

where $\tilde{A} = A + BK + G_{\mathbf{d}}$ such that $\tilde{A}^\top P + P\tilde{A} \preceq 0$ and

$$\tilde{F}(y) = \begin{cases} \left\{\frac{y}{\|y\|}\right\} & \text{if} \quad y \neq 0 \\ B & \text{if} \quad y = 0 \end{cases}$$

where B is the unit ball in $\mathbb{R}^n$ with the norm $\|y\| = \sqrt{y^\top Py}$.

Let us denote $q_{i+1} = \|y_{i+1}\|$ and $z_{i+1} = \frac{y_{i+1}}{\|y_{i+1}\|}$ then the inclusion (22) has the following solution

- if $y_i^\top (I_n - \tilde{A})^{-\top} P (I_n - \tilde{A})^{-1} y_i \leq h^2$ then

$$q_{i+1} = 0 \quad \text{and} \quad z_{i+1} = h^{-1}(I_n - h\tilde{A})^{-1} y_i;$$

- otherwise, q_{i+1} and z_{i+1} are derived as solution to

$$\left((q_{i+1} + h)I_n - h\tilde{A}\right) z_{i+1} = y_i, \quad z_{i+1}^\top P z_{i+1} = 1. \quad (23)$$

Solution to (23) always exists due to Theorem 1. To find it the equation

$$y_i^\top \left((q_{i+1} + h)I_n - h\tilde{A}\right)^{-\top} P \left((q_{i+1} + h)I_n - h\tilde{A}\right)^{-1} y_i = 1$$

that is polynomial with respect to q_{i+1}, must be initially solved.

Taking into account $x = \mathbf{d}(\ln\|y\|)\frac{y}{\|y\|}$ and $u(\mathbf{d}(s)x) = u(x)$ we derive that the implicit discretization of the control is given by

$$u(x_{i+1}) = u(z_{i+1}) = Kz_{i+1}.$$

According to the conventional implicit discretization technique, this control value is suggested to be selected for the time interval $[t_i, t_{i+1})$ during digital implementation of sliding mode control.

For $n = 2$ the system (23) implies quartic equation with respect to q_{i+1}, so it can be solved explicitly using Ferrari formulas. The simulation results for $n = 2$, $h = 0.02$,

$$P = \begin{pmatrix} 9.1050 & 1.7829 \\ 1.7829 & 0.8914 \end{pmatrix}, K = \begin{pmatrix} -10.2139 & -3.0000 \end{pmatrix},$$

are given in Fig. 2 and Fig. 3. They confirm finite-time convergence of $\{x_i\}$ to zero in a finite-number of steps, where $x_i = \mathbf{d}(\ln \|y_i\|)\frac{y_i}{\|y_i\|}$ and $\{y_i\}$ is the solution to (22).

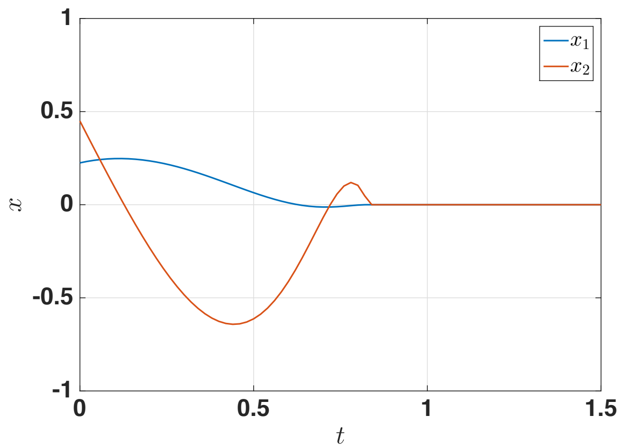

Fig. 2. System states

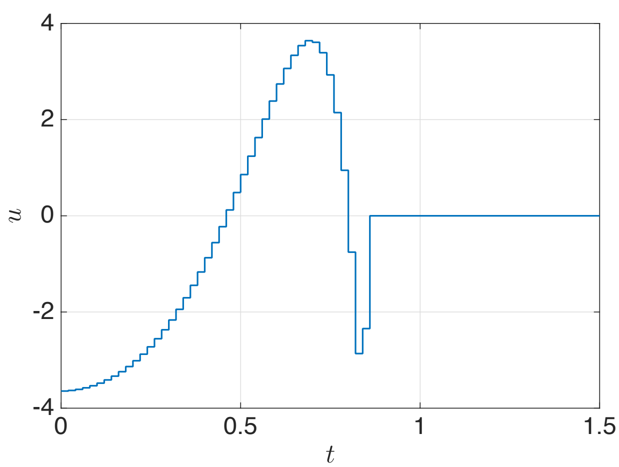

Fig. 3. Control input

VI. Discussions and Conclusions

The problem of consistent discrete-time approximation of finite-time stable systems is studied. It is shown that any generalized homogeneous finite-time stable system admits implicit Euler discretization preserving finite-time convergence. The topological equivalence [29] of homogeneous stable system to a quadratically stable one is utilized for the design of this scheme. Theoretical results are supported with numerical simulations.

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